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Eleven-Dimensional Gauge Theory for the M Algebra as an Abelian Semigroup Expansion of $\mathfrak{osp}(32|1)$

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Abstract

An eleven-dimensional gauge theory for the M Algebra is put forward. The gauge-invariant lagrangian belongs to the class of transgression lagrangians, which modify Chern–Simons theory with the addition of a regularizing boundary term.

The M Algebra-invariant tensor needed in order to write down the transgression lagrangian comes from regarding the Algebra as an Abelian Semigroup Expansion of the orthosymplectic algebra $\mathfrak{osp}(32|1)$. The lagrangian is displayed in an explicitly Lorentz-invariant way by means of a transgression-specific subspace separation method based on the extended Cartan homotopy formula.

The lower-dimensional dynamics produced by the theory is shown to be tightly constrained, but allowing for nonzero torsion might help break the chains. Symmetrical boundary conditions directly derived from the action are considered, and some alternatives to solve them are provided. We also comment on a possible physical interpretation of the two-connection setting inherent to any transgression gauge field theory.

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I. INTRODUCTION

String Theory and eleven-dimensional Supergravity became inextricably linked after the arrival of the M-Theory Paradigm. All efforts notwithstanding, the low-energy regime of M Theory remains better known than its non-perturbative description. However, the possibility has been pointed out that M Theory may be non-perturbatively related to, or even formulated as, an eleven-dimensional Chern–Simons theory [1, 2, 3].

Chern–Simons (CS) Theory has quite compelling features. On one hand, it belongs to the restricted class of gauge field theories, with a one-form gauge connection as the sole dynamical field. On the other hand, and in contrast with usual Yang–Mills theory, there's no *a priori* metric needed to define the CS lagrangian, so that the theory turns out to be background-free. CS Supergravities (see, e.g., [4] and references therein) exist in every odd dimension; three-dimensional General Relativity was famously quantized by making the connection to CS [5].

There are also a couple of issues regarding CS systems. Most importantly, the CS lagrangian is not fully gauge-invariant, but changes by a closed form under gauge transformations. This means that boundary conditions and Noether charges are intrinsically ambiguous, as there is no symmetry principle that can rule out the addition of an arbitrary exact form to the lagrangian.

Or maybe there is. Transgression forms [16, 17, 18, 19, 20, 21] are the matrix where CS forms stem from. The main difference between CS and Transgression forms concerns a new, regularizing boundary term which renders the Transgression form fully gauge invariant. As a consequence, the boundary conditions and Noether charges computed from a transgression action have the chance to be physically meaningful.

A transgression gauge field theory for the M Algebra may take us one step closer to understanding the non-perturbative description of M Theory.

The paper is organized as follows. In sec. II the derivation of the M Algebra as an Abelian Semigroup Expansion of $\mathfrak{osp}\,(32|1)$ is performed, and a recipe for an M Algebra-invariant tensor is given. Section III presents the transgression lagrangian. This section borrows results from Refs. [18, 21] in order to write the lagrangian in an explicitly Lorentz-invariant way. In sec. IV we comment on the dynamics produced by the transgression lagrangian. A discussion on symmetric boundary conditions is also included. We close with the conclusions and some final remarks in sec. V.

II. THE M ALGEBRA AS AN S-EXPANSION OF $\mathfrak{osp}\,(32|1)$

There are several procedures to obtain new Lie algebras from a given one, such as contraction, deformation, extension and expansion of algebras. Among these methods, the *expansion* [6] is the only one that is not dimension-preserving: in general, it leads to algebras with a dimensionality higher than the original one.

As an important example stands the M Algebra [6, 7], which, with its 583 bosonic generators, can be regarded as an expansion [27] of the orthosymplectic algebra $\mathfrak{osp}\,(32|1)$, which has only 528 (both algebras have the same number of fermionic generators).

In a nutshell, the expansion method can be described as follows. Consider the original algebra as described by its associated Maurer–Cartan forms on the group manifold. Some of the group parameters are rescaled by a factor λ, and the Maurer–Cartan forms are expanded as a power series in λ. This series is finally truncated in a way that assures the closure of the expanded algebra. The subject is thoroughly treated by de Azcárraga and Izquierdo in Ref. [8] and de Azcárraga, Izquierdo, Picón and Varela in [6].

In Ref. [10] a natural outgrowth of the expansion method was proposed, which involves the use of an arbitrary, discrete abelian semigroup S [28]. This *Abelian Semigroup Expansion* method, 'S-expansion' for short, reproduces the results of the Maurer–Cartan forms power-series expansion for a particular choice of the semigroup S, but is formulated using the Lie algebra generators rather than the associated Maurer–Cartan forms. For this reason, in the S-expansion context it becomes very clear what the structure of an invariant tensor for the expanded algebra should be, and therefore, this scheme turns out to be especially suitable for the construction of Chern–Simons (CS) and Transgression forms for the expanded algebra.

In the next sections we analyze the construction of the M Algebra as an S-expansion of $\mathfrak{osp}\,(32|1)$.

A. The S-Expansion Procedure

In this section we briefly review the general Abelian Semigroup Expansion Procedure. We shall not attempt to cover here the S-expansion method in full, as this would take us far afield from our present subject. We refer the interested reader to the literature [10].

Consider a Lie algebra $\mathfrak{g}$ and an abelian semigroup $S = \{\lambda_\alpha\}$. According to Theorem 1 from Ref. [10], the direct product $S \otimes \mathfrak{g}$ is also a Lie algebra. Interestingly, there are cases when it is possible to systematically extract subalgebras from $S \otimes \mathfrak{g}$. Start by decomposing $\mathfrak{g}$ in a direct sum of subspaces, as in

$$\mathfrak{g} = \bigoplus_{p \in I} V_p, \tag{1}$$

where I is a set of indices. The internal structure of $\mathfrak{g}$ can be codified through the subsets

$i_{(p,q)} \subset I$ according to

$$[V_p, V_q] \subset \bigoplus_{r \in i_{(p,q)}} V_r. \tag{2}$$

When the semigroup S can be decomposed in subsets S_p, $S = \bigcup_{p\in I} S_p$, such that they satisfy the condition

$$S_p \times S_q \subset \bigcap_{r \in i_{(p,q)}} S_r, \tag{3}$$

then we have that

$$\mathfrak{G}_{\mathrm{R}} = \bigoplus_{p \in I} S_p \otimes V_p \tag{4}$$

is a 'resonant subalgebra' of $S \otimes \mathfrak{g}$ (see Theorem 2 from Ref. [10]).

An even smaller algebra can be obtained when there is a zero element in the semigroup, i.e., an element $0_S \in S$ such that, for all $\lambda_\alpha \in S$, $0_S \lambda_\alpha = 0_S$. When this is the case, the whole $0_S \otimes \mathfrak{g}$ sector can be removed from the resonant subalgebra by imposing $0_S \otimes \mathfrak{g} = 0$. The remaining piece, dubbed 0_S-forced algebra, continues to be a Lie algebra (see 0_S-forcing and Theorem 3 from Ref. [10]).

In the next section these mathematical tools will be used in order to show how the M Algebra can be constructed from $\mathfrak{osp}\,(32|1)$.

B. M Algebra as an S-expansion

In this section we roughly sketch the steps to be undertaken in order to obtain the M Algebra as an S-Expansion of $\mathfrak{osp}\,(32|1)$.

As with any expansion, the first step consists of splitting the algebra $\mathfrak{osp}\,(32|1)$ in distinct subspaces. This is accomplished by defining

$$V_0 = \left\{ \boldsymbol{J}_{ab}^{(\mathfrak{osp})} \right\}, \tag{5}$$

$$V_1 = \left\{ \boldsymbol{Q}^{(\mathfrak{osp})} \right\}, \tag{6}$$

$$V_2 = \left\{ \boldsymbol{P}_a^{(\mathfrak{osp})}, \boldsymbol{Z}_{a_1 \cdots a_5}^{(\mathfrak{osp})} \right\}. \tag{7}$$

Here V_0 corresponds to the Lorentz Algebra, V_1 to the fermions and V_2 to the remaining bosonic generators, namely AdS boosts and the M5-brane piece. The algebraic structure satisfied by these subspaces is common to every superalgebra, as can be seen from the equations

$$[V_0, V_0] \subset V_0, \tag{8}$$

$$[V_0, V_1] \subset V_1, \tag{9}$$

$$[V_0, V_2] \subset V_2, \tag{10}$$

$$[V_1, V_1] \subset V_0 \oplus V_2, \tag{11}$$

$$[V_1, V_2] \subset V_1, \tag{12}$$

$$[V_2, V_2] \subset V_0 \oplus V_2. \tag{13}$$

The second step is particular to the method of S-expansions, and deals with finding an abelian semigroup S which can be partitioned in a 'resonant' way with respect to (8)–(13). This semigroup exists and is given by $S_{\mathrm{E}}^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$, with the defining product

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, \text{ when } \alpha + \beta \leq 2, \\ \lambda_3, \text{ otherwise.} \end{cases} \tag{14}$$

A straightforward but important observation is that, for each $\lambda_\alpha \in S_{\mathrm{E}}^{(2)}$, $\lambda_3 \lambda_\alpha = \lambda_3$, so that λ_3 plays the rôle of the zero element inside $S_{\mathrm{E}}^{(2)}$.

Consider now the partition $S_{\mathrm{E}}^{(2)} = S_0 \cup S_1 \cup S_2$, with

$$S_0 = \{\lambda_0, \lambda_2, \lambda_3\}, \tag{15}$$

$$S_1 = \{\lambda_1, \lambda_3\}, \tag{16}$$

$$S_2 = \{\lambda_2, \lambda_3\}. \tag{17}$$

This partition is said to be resonant, since it satisfies [compare eqs. (8)–(13) with eqs. (18)–(23)]

$$S_0 \times S_0 \subset S_0, \tag{18}$$

$$S_0 \times S_1 \subset S_1, \tag{19}$$

$$S_0 \times S_2 \subset S_2, \tag{20}$$

$$S_1 \times S_1 \subset S_0 \cap S_2, \tag{21}$$

$$S_1 \times S_2 \subset S_1, \tag{22}$$

$$S_2 \times S_2 \subset S_0 \cap S_2. \tag{23}$$

TABLE I: The M Algebra can be regarded as an $S_{\mathrm{E}}^{(2)}$-Expansion of $\mathfrak{osp}\,(32|1)$. The table shows the relation between generators from both algebras. The three levels correspond to the three columns in Fig. 1 or, alternatively, to the three subsets into which $S_{\mathrm{E}}^{(2)}$ has been partitioned.

$\mathfrak{G}_{\mathrm{R}}$ Subspaces	Generators
$S_0 \otimes V_0$	$\boldsymbol{J}_{ab} = \lambda_0 \boldsymbol{J}_{ab}^{(\mathfrak{osp})}$
	$\boldsymbol{Z}_{ab} = \lambda_2 \boldsymbol{J}_{ab}^{(\mathfrak{osp})}$
	$0 = \lambda_3 \boldsymbol{J}_{ab}^{(\mathfrak{osp})}$
$S_1 \otimes V_1$	$\boldsymbol{Q} = \lambda_1 \boldsymbol{Q}^{(\mathfrak{osp})}$
	$0 = \lambda_3 \boldsymbol{Q}^{(\mathfrak{osp})}$
$S_2 \otimes V_2$	$\boldsymbol{P}_a = \lambda_2 \boldsymbol{P}_a^{(\mathfrak{osp})}$
	$0 = \lambda_3 \boldsymbol{P}_a^{(\mathfrak{osp})}$
	$\boldsymbol{Z}_{abcde} = \lambda_2 \boldsymbol{Z}_{abcde}^{(\mathfrak{osp})}$
	$0 = \lambda_3 \boldsymbol{Z}_{abcde}^{(\mathfrak{osp})}$

Theorem 2 from Ref. [10] now assures us that

$$\mathfrak{G}_{\mathrm{R}} = (S_0 \otimes V_0) \oplus (S_1 \otimes V_1) \oplus (S_2 \otimes V_2) \tag{24}$$

is a *resonant subalgebra* of $S_{\mathrm{E}}^{(2)} \otimes \mathfrak{g}$.

As a last step, impose the condition $\lambda_3 \otimes \mathfrak{g} = 0$ on $\mathfrak{G}_{\mathrm{R}}$ and relabel its generators as in Table I. This procedure gives us the M Algebra, whose (anti)commutation relations read

$$\left[\boldsymbol{J}^{ab}, \boldsymbol{J}_{cd}\right] = \delta_{ecd}^{abf} \boldsymbol{J}^{e}{}_{f}, \tag{25}$$

$$\left[\boldsymbol{J}^{ab}, \boldsymbol{P}_{c}\right] = \delta_{ec}^{ab} \boldsymbol{P}^{e}, \tag{26}$$

$$\left[\boldsymbol{J}^{ab}, \boldsymbol{Z}_{cd}\right] = \delta_{ecd}^{abf} \boldsymbol{Z}^{e}{}_{f}, \tag{27}$$

$$\left[\boldsymbol{J}^{ab}, \boldsymbol{Z}_{c_1 \cdots c_5}\right] = \frac{1}{4!} \delta_{dc_1 \cdots c_5}^{abe_1 \cdots e_4} \boldsymbol{Z}^{d}{}_{e_1 \cdots e_4}, \tag{28}$$

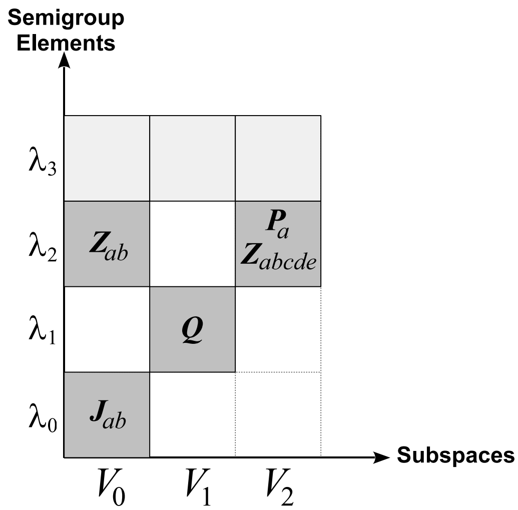

FIG. 1: The shaded region denotes the resonant subalgebra. Dark shaded areas correspond to M Algebra itself and light-gray areas correspond to the $\lambda_3 \otimes \mathfrak{osp}\,(32|1)$ sector.

$$\begin{aligned}
[\boldsymbol{P}_a, \boldsymbol{P}_b] &= 0, && (29)\\
[\boldsymbol{P}_a, \boldsymbol{Z}_{bc}] &= 0, && (30)\\
[\boldsymbol{P}_a, \boldsymbol{Z}_{b_1\cdots b_5}] &= 0, && (31)\\
[\boldsymbol{Z}_{a_1a_2}, \boldsymbol{Z}_{b_1b_2}] &= 0, && (32)\\
[\boldsymbol{Z}_{a_1a_2}, \boldsymbol{Z}_{b_1\cdots b_5}] &= 0, && (33)\\
[\boldsymbol{Z}_{a_1\cdots a_5}, \boldsymbol{Z}_{b_1\cdots b_5}] &= 0, && (34)
\end{aligned}$$

$$[\boldsymbol{J}_{ab}, \boldsymbol{Q}] = -\frac{1}{2}\Gamma_{ab}\boldsymbol{Q}, \tag{35}$$

$$\begin{aligned}
[\boldsymbol{P}_a, \boldsymbol{Q}] &= 0, && (36)\\
[\boldsymbol{Z}_{ab}, \boldsymbol{Q}] &= 0, && (37)\\
[\boldsymbol{Z}_{abcde}, \boldsymbol{Q}] &= 0, && (38)
\end{aligned}$$

$$\left\{\boldsymbol{Q}, \bar{\boldsymbol{Q}}\right\} = \frac{1}{8}\left(\Gamma^a\boldsymbol{P}_a - \frac{1}{2}\Gamma^{ab}\boldsymbol{Z}_{ab} + \frac{1}{5!}\Gamma^{abcde}\boldsymbol{Z}_{abcde}\right). \tag{39}$$

A clearer picture of the Algebra's structure can be obtained from the diagram in Fig. 1. The subspaces of $\mathfrak{osp}\,(32|1)$ are represented on the horizontal axis, and the semigroup elements on the vertical one. The whole shaded region corresponds to the resonant subalgebra, and the light gray one to the $\lambda_3 \otimes \mathfrak{osp}\,(32|1)$ sector, which is mapped to zero. The dark gray sector corresponds to M Algebra itself. The diagram allows us to graphically encode the subset partition (15)–(17) on each column, and makes checking the closure of the algebra a straightforward matter.

Large sectors of the resonant subalgebra are abelianized after imposing the condition $\lambda_3 \otimes \mathfrak{osp}\,(32|1) = 0$. This condition also plays a fundamental rôle in the shaping of the invariant tensor for the M Algebra as an S-expansion of $\mathfrak{osp}\,(32|1)$. In this way, its effects are felt all the way down to the theory's specific dynamic properties.

C. M-Algebra Invariant Tensor

Finding all possible invariant tensors for an arbitrary algebra remains, to the best of our knowledge, as an important open problem. Nevertheless, once a matrix representation for a Lie algebra is known, the (super)trace always provides with an invariant tensor. But precisely in our case, this is not a wise choice: in general, it is possible to prove that when the condition $0_S \otimes \mathfrak{g} = 0$ is imposed, the supertrace for the S-expanded algebra generators will correspond to just a very small piece of the whole (super)trace for the $\mathfrak{g}$-generators. For the particular case of the M Algebra, the only non-vanishing component of the supertrace is $\mathrm{Tr}\,(\boldsymbol{J}_{a_1 b_1} \cdots \boldsymbol{J}_{a_n b_n})$. A CS Lagrangian constructed with this invariant tensor would lead to an 'exotic gravity', where the fermions, the central charges and even the vielbein would be absent from the invariant tensor. For this reason, it becomes a necessity to work out other kinds of invariant tensors; very interesting work on precisely this point has been developed in Refs. [13, 15], where an invariant tensor for the M Algebra is obtained from the Noether method, finally leading to a CS M-Algebra Supergravity in eleven dimensions.

In the context of an S-expansion, Theorems 4 and 5 from Ref. [10] provide with non-trivial invariant tensors different from the supertrace.

Let $\lambda_{\alpha_1}, \ldots, \lambda_{\alpha_n} \in S$ be arbitrary elements of the semigroup S. Their product can be written as

$$\lambda_{\alpha_1} \cdots \lambda_{\alpha_n} = \lambda_{\gamma(\alpha_1,\ldots,\alpha_n)}. \tag{40}$$

This product law can be conveniently encoded by the n-selector $K_{\alpha_1 \cdots \alpha_n}{}^{\rho}$, which is defined as

$$K_{\alpha_1\cdots\alpha_n}{}^{\rho} = \begin{cases} 1, \text{ when } \rho = \gamma\left(\alpha_1, \ldots, \alpha_n\right) \\ 0, \text{ otherwise.} \end{cases} \tag{41}$$

Theorem 4 from Ref. [10] states that

$$\left\langle \boldsymbol{T}_{(A_1,\alpha_1)} \cdots \boldsymbol{T}_{(A_n,\alpha_n)} \right\rangle = \alpha_\gamma K_{\alpha_1\cdots\alpha_n}{}^{\gamma} \left\langle \boldsymbol{T}_{A_1} \cdots \boldsymbol{T}_{A_n} \right\rangle \tag{42}$$

corresponds to an invariant tensor for the S-expanded algebra without 0_S-forcing, where α_γ are arbitrary constants.

When the semigroup contains a zero element $0_S \in S$, a smaller algebra can be obtained by '0_S-forcing' the S-expanded algebra, i.e., by mapping all elements of the form $0_S \otimes \mathfrak{g}$ to zero. Writing λ_i for the nonzero elements of S, Theorem 5 from Ref. [10] assures that

$$\left\langle \boldsymbol{T}_{(A_1,i_1)} \cdots \boldsymbol{T}_{(A_n,i_n)} \right\rangle = \alpha_j K_{i_1\cdots i_n}{}^{j} \left\langle \boldsymbol{T}_{A_1} \cdots \boldsymbol{T}_{A_n} \right\rangle \tag{43}$$

is an invariant tensor for the 0_S-forced algebra, with α_j being arbitrary constants. As can be seen by comparing eq. (42) with eq. (43), this invariant tensor corresponds to a 'pruning' of (42).

In the M-Algebra case, one must compute the components of $K_{i_1\cdots i_6}{}^{j}$ for $S_{\mathrm{E}}^{(2)}$. Using the multiplication law (14), these are easily seen to be

$$K_{i_1\cdots i_6}{}^{j} = \delta^{j}_{(i_1+\cdots+i_6)}, \tag{44}$$

where the δ is the Kronecker delta.

Using eqs. (43) and (44), we have that the *only* non-vanishing components of the M Algebra-invariant tensor are given by

$$\left\langle \boldsymbol{J}_{a_1b_1} \cdots \boldsymbol{J}_{a_6b_6} \right\rangle_{\mathrm{M}} = \alpha_0 \left\langle \boldsymbol{J}_{a_1b_1} \cdots \boldsymbol{J}_{a_6b_6} \right\rangle_{\mathfrak{osp}}, \tag{45}$$

$$\left\langle \boldsymbol{J}_{a_1b_1} \cdots \boldsymbol{J}_{a_5b_5} \boldsymbol{P}_c \right\rangle_{\mathrm{M}} = \alpha_2 \left\langle \boldsymbol{J}_{a_1b_1} \cdots \boldsymbol{J}_{a_5b_5} \boldsymbol{P}_c \right\rangle_{\mathfrak{osp}}, \tag{46}$$

$$\left\langle \boldsymbol{J}_{a_1b_1} \cdots \boldsymbol{J}_{a_5b_5} \boldsymbol{Z}_{a_6b_6} \right\rangle_{\mathrm{M}} = \alpha_2 \left\langle \boldsymbol{J}_{a_1b_1} \cdots \boldsymbol{J}_{a_6b_6} \right\rangle_{\mathfrak{osp}}, \tag{47}$$

$$\left\langle \boldsymbol{J}_{a_1b_1} \cdots \boldsymbol{J}_{a_5b_5} \boldsymbol{Z}_{c_1\cdots c_5} \right\rangle_{\mathrm{M}} = \alpha_2 \left\langle \boldsymbol{J}_{a_1b_1} \cdots \boldsymbol{J}_{a_5b_5} \boldsymbol{Z}_{c_1\cdots c_5} \right\rangle_{\mathfrak{osp}}, \tag{48}$$

$$\left\langle \boldsymbol{Q} \boldsymbol{J}_{a_1b_1} \cdots \boldsymbol{J}_{a_4b_4} \bar{\boldsymbol{Q}} \right\rangle_{\mathrm{M}} = \alpha_2 \left\langle \boldsymbol{Q} \boldsymbol{J}_{a_1b_1} \cdots \boldsymbol{J}_{a_4b_4} \bar{\boldsymbol{Q}} \right\rangle_{\mathfrak{osp}}, \tag{49}$$

where α_0 and α_2 are arbitrary constants.

It is noteworthy that this invariant tensor for the M Algebra, even if much bigger than the supertrace [which would consist of (45) alone], is still a lot smaller than the one for

$\mathfrak{osp}\,(32|1)$. This is a common feature of 0_S-forced algebras. In stark contrast, S-expanded algebras which do not arise from a 0_S-forcing process do have invariant tensors bigger than the one for the original algebra. This fact shapes the dynamics of the theory to a great extent, as we shall see in section IV.

The supersymmetrized supertrace will be used to provide an invariant tensor for $\mathfrak{osp}\,(32|1)$, with the 32×32 Dirac matrices in eleven dimensions as a matrix representation for the bosonic subalgebra, $\mathfrak{sp}\,(32)$. The representation with $\Gamma_1\cdots\Gamma_{11}=+\mathbb{1}$ was chosen. In order to write the lagrangian, field equations and boundary conditions, it is very useful to have the components of the $\mathfrak{osp}\,(32|1)$-invariant tensor with its indices contracted with arbitrary tensors. An explicit calculation gives us

$$L_1^{a_1b_1}\cdots L_5^{a_5b_5}B_1^c\left\langle \boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_5b_5}\boldsymbol{P}_c\right\rangle_{\mathfrak{osp}}=\frac{1}{2}\varepsilon_{a_1\cdots a_{11}}L_1^{a_1a_2}\cdots L_5^{a_9a_{10}}B_1^{a_{11}}, \tag{50}$$

$$\begin{aligned}
&L_1^{a_1b_1}\cdots L_6^{a_6b_6}\left\langle \boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_6b_6}\right\rangle_{\mathfrak{osp}}\\
=\;&\frac{1}{3}\sum_{\sigma\in S_6}\left[\frac{1}{4}\operatorname{Tr}\left(L_{\sigma(1)}L_{\sigma(2)}\right)\operatorname{Tr}\left(L_{\sigma(3)}L_{\sigma(4)}\right)\operatorname{Tr}\left(L_{\sigma(5)}L_{\sigma(6)}\right)+\right.\\
&-\operatorname{Tr}\left(L_{\sigma(1)}L_{\sigma(2)}L_{\sigma(3)}L_{\sigma(4)}\right)\operatorname{Tr}\left(L_{\sigma(5)}L_{\sigma(6)}\right)+\\
&\left.+\frac{16}{15}\operatorname{Tr}\left(L_{\sigma(1)}L_{\sigma(2)}L_{\sigma(3)}L_{\sigma(4)}L_{\sigma(5)}L_{\sigma(6)}\right)\right],
\end{aligned} \tag{51}$$

$$\begin{aligned}
&L_1^{a_1b_1}\cdots L_5^{a_5b_5}B_5^{c_1\cdots c_5}\left\langle \boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_5b_5}\boldsymbol{Z}_{c_1\cdots c_5}\right\rangle_{\mathfrak{osp}}\\
=\;&\frac{1}{3}\varepsilon_{a_1\cdots a_{11}}\sum_{\sigma\in S_5}\left[-\frac{5}{4}L_{\sigma(1)}^{a_1a_2}\cdots L_{\sigma(4)}^{a_7a_8}\left[L_{\sigma(5)}\right]_{bc}B_5^{bca_9a_{10}a_{11}}+\right.\\
&+10L_{\sigma(1)}^{a_1a_2}L_{\sigma(2)}^{a_3a_4}L_{\sigma(3)}^{a_5a_6}\left[L_{\sigma(4)}\right]^{a_7}{}_b\left[L_{\sigma(5)}\right]^{a_8}{}_cB_5^{bca_9a_{10}a_{11}}+\\
&+\frac{1}{4}L_{\sigma(1)}^{a_1a_2}L_{\sigma(2)}^{a_3a_4}L_{\sigma(3)}^{a_5a_6}B_5^{a_7\cdots a_{11}}\operatorname{Tr}\left(L_{\sigma(4)}L_{\sigma(5)}\right)+\\
&\left.-L_{\sigma(1)}^{a_1a_2}L_{\sigma(2)}^{a_3a_4}\left[L_{\sigma(3)}L_{\sigma(4)}L_{\sigma(5)}\right]^{a_5a_6}B_5^{a_7\cdots a_{11}}\right],
\end{aligned} \tag{52}$$

$$\begin{aligned}
&L_1^{a_1b_1}\cdots L_4^{a_4b_4}\bar{\chi}_\alpha\zeta^\beta\left\langle \boldsymbol{Q}^\alpha\boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_4b_4}\bar{\boldsymbol{Q}}_\beta\right\rangle_{\mathfrak{osp}}\\
=\;&-\frac{1}{240}\varepsilon_{a_1\cdots a_8abc}L_1^{a_1a_2}\cdots L_4^{a_7a_8}\bar{\chi}\Gamma^{abc}\zeta+\\
&+\frac{1}{60}\sum_{\sigma\in S_4}\left[\frac{3}{4}\operatorname{Tr}\left(L_{\sigma(1)}L_{\sigma(2)}\right)L_{\sigma(3)}^{a_1a_2}L_{\sigma(4)}^{a_3a_4}\bar{\chi}\Gamma_{a_1\cdots a_4}\zeta+\right.\\
&-2L_{\sigma(1)}^{a_1a_2}\left[L_{\sigma(2)}L_{\sigma(3)}L_{\sigma(4)}\right]^{a_3a_4}\bar{\chi}\Gamma_{a_1\cdots a_4}\zeta+\\
&+\frac{3}{4}\operatorname{Tr}\left(L_{\sigma(1)}L_{\sigma(2)}\right)\operatorname{Tr}\left(L_{\sigma(3)}L_{\sigma(4)}\right)\bar{\chi}\zeta+\\
&\left.-\operatorname{Tr}\left(L_{\sigma(1)}L_{\sigma(2)}L_{\sigma(3)}L_{\sigma(4)}\right)\bar{\chi}\zeta\right],
\end{aligned} \tag{53}$$

where Tr stands for the trace in the Lorentz indices, i.e. $\mathrm{Tr}\left(L_i L_j\right) = \left(L_i\right)^a{}_b \left(L_j\right)^b{}_a$.

III. THE M-ALGEBRA LAGRANGIAN

A. Transgression Gauge Field Theory for the M Algebra

The gauge invariant lagrangian we shall use depends on two M Algebra-valued, one-form gauge connections $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$. These fields can be written as

$$\boldsymbol{A} = \boldsymbol{\omega} + \boldsymbol{e} + \boldsymbol{b}_2 + \boldsymbol{b}_5 + \bar{\boldsymbol{\psi}}, \tag{54}$$

$$\bar{\boldsymbol{A}} = \bar{\boldsymbol{\omega}} + \bar{\boldsymbol{e}} + \bar{\boldsymbol{b}}_2 + \bar{\boldsymbol{b}}_5 + \bar{\boldsymbol{\chi}}, \tag{55}$$

where each term takes values on a different subspace of the M Algebra, namely

$$\boldsymbol{\omega} = \frac{1}{2}\omega^{ab}\boldsymbol{J}_{ab}, \tag{56}$$

$$\boldsymbol{e} = e^a \boldsymbol{P}_a, \tag{57}$$

$$\boldsymbol{b}_2 = \frac{1}{2} b_2^{ab} \boldsymbol{Z}_{ab}, \tag{58}$$

$$\boldsymbol{b}_5 = \frac{1}{5!} b_5^{abcde} \boldsymbol{Z}_{abcde}, \tag{59}$$

$$\bar{\boldsymbol{\psi}} = \bar{\psi}_\alpha \boldsymbol{Q}^\alpha, \tag{60}$$

and similarly for $\bar{\boldsymbol{A}}$. The curvature for $\boldsymbol{A}$ reads

$$\boldsymbol{F} = \boldsymbol{R} + \boldsymbol{F}_P + \boldsymbol{F}_2 + \boldsymbol{F}_5 + \mathrm{D}_\omega \bar{\boldsymbol{\psi}}, \tag{61}$$

where again each term refers to a different subspace:

$$\boldsymbol{R} = \frac{1}{2} R^{ab} \boldsymbol{J}_{ab}, \tag{62}$$

$$\boldsymbol{F}_P = \left(T^a + \frac{1}{16}\bar{\psi}\Gamma^a\psi\right)\boldsymbol{P}_a, \tag{63}$$

$$\boldsymbol{F}_2 = \frac{1}{2}\left(\mathrm{D}_\omega b^{ab} - \frac{1}{16}\bar{\psi}\Gamma^{ab}\psi\right)\boldsymbol{Z}_{ab}, \tag{64}$$

$$\boldsymbol{F}_5 = \frac{1}{5!}\left(\mathrm{D}_\omega b^{a_1\cdots a_5} + \frac{1}{16}\bar{\psi}\Gamma^{a_1\cdots a_5}\psi\right)\boldsymbol{Z}_{a_1\cdots a_5}, \tag{65}$$

$$\mathrm{D}_\omega \bar{\boldsymbol{\psi}} = \mathrm{D}_\omega \bar{\psi}\boldsymbol{Q}. \tag{66}$$

The Lorentz covariant derivative for spinors has the usual form,

$$\mathrm{D}_{\omega}\psi = \mathrm{d}\psi + \frac{1}{4}\omega^{ab}\Gamma_{ab}\psi, \tag{67}$$

$$\mathrm{D}_{\omega}\bar{\psi} = \mathrm{d}\bar{\psi} - \frac{1}{4}\omega^{ab}\bar{\psi}\Gamma_{ab}. \tag{68}$$

As lagrangian for this gauge theory we shall take

$$L_{\mathrm{T}}^{(11)}\left(\boldsymbol{A}, \bar{\boldsymbol{A}}\right) = Q_{\boldsymbol{A}\leftarrow\bar{\boldsymbol{A}}}^{(11)} = 6\int_{0}^{1}\mathrm{d}t\left\langle \boldsymbol{\Theta}\boldsymbol{F}_{t}^{5}\right\rangle_{\mathrm{M}}, \tag{69}$$

where

$$\boldsymbol{\Theta} = \boldsymbol{A} - \bar{\boldsymbol{A}}, \tag{70}$$

$$\boldsymbol{A}_{t} = \bar{\boldsymbol{A}} + t\boldsymbol{\Theta}, \tag{71}$$

$$\boldsymbol{F}_{t} = \mathrm{d}\boldsymbol{A}_{t} + \boldsymbol{A}_{t}^{2}. \tag{72}$$

The lagrangian (69) corresponds to a transgression form [16, 17, 18, 19, 20, 21]. Transgression forms are intimately related to CS forms, since they can be written as the difference of two CS forms plus a boundary term. The presence of this crucial boundary term cures some pathologies present in standard CS Theory, such as ill-defined conserved charges [20].

The general form of the lagrangian given in eq. (69) suffices in order to derive field equations, boundary conditions and Noether charges. Nevertheless, an explicit version is highly desirable because it clearly shows the physical content of the theory; in particular, a separation in bulk and boundary contributions is essential. This important task can be painstakingly long if approached naïvely, i.e. through the sole use of Leibniz's rule. A way out of the bog is provided by the subspace separation method presented in Refs. [18, 21]. This method serves a double purpose; on one hand, it splits the lagrangian in bulk and boundary terms and, on the other, it allows the separation of the bulk lagrangian in reflection of the algebra's subspace structure.

The subspace separation method is based on the iterative use of the 'Triangle Equation'

$$Q_{\boldsymbol{A}\leftarrow\bar{\boldsymbol{A}}}^{(2n+1)} = Q_{\boldsymbol{A}\leftarrow\tilde{\boldsymbol{A}}}^{(2n+1)} + Q_{\tilde{\boldsymbol{A}}\leftarrow\bar{\boldsymbol{A}}}^{(2n+1)} + \mathrm{d}Q_{\boldsymbol{A}\leftarrow\tilde{\boldsymbol{A}}\leftarrow\bar{\boldsymbol{A}}}^{(2n)}, \tag{73}$$

which itself is a particular case of the extended Cartan homotopy formula [22]. A first outcome of eq. (73) is the proof of the above-mentioned relation between transgression and CS forms; setting $\tilde{\boldsymbol{A}} = 0$ one readily finds

$$Q_{\boldsymbol{A}\leftarrow\bar{\boldsymbol{A}}}^{(2n+1)} = Q_{\boldsymbol{A}\leftarrow 0}^{(2n+1)} - Q_{\bar{\boldsymbol{A}}\leftarrow 0}^{(2n+1)} + \mathrm{d}Q_{\boldsymbol{A}\leftarrow 0\leftarrow\bar{\boldsymbol{A}}}^{(2n)}, \tag{74}$$

where the first two terms correspond to CS forms. In secs. III C and III D the subspace separation method is applied to the lagrangian (69).

Under infinitesimal variations $\boldsymbol{A} \to \boldsymbol{A} + \delta\boldsymbol{A}$, $\bar{\boldsymbol{A}} \to \bar{\boldsymbol{A}} + \delta\bar{\boldsymbol{A}}$, the lagrangian (69) changes by

$$\delta L_{\mathrm{T}}^{(11)} = 6\left(\left\langle \delta\boldsymbol{A}\boldsymbol{F}^5\right\rangle - \left\langle \delta\bar{\boldsymbol{A}}\bar{\boldsymbol{F}}^5\right\rangle\right) + 30\mathrm{d}\int_0^1 \mathrm{d}t \left\langle \delta\boldsymbol{A}_t\boldsymbol{\theta}\boldsymbol{F}_t^4\right\rangle . \tag{75}$$

From eq. (75) one can read off both the field equations (first term) and the boundary conditions (second term). These are analyzed in more detail in sec. IV.

B. Theory Doubling

There are several interesting issues concerning this choice of lagrangian. As signaled in (69), $L_{\mathrm{T}}^{(11)}$ depends on the two M Algebra-valued connections $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$. This means that the field content is doubled as compared with standard CS Theory, where $\bar{\boldsymbol{A}} = 0$ from the outset.

Despite its mathematical appeal, the presence of two connections may seem untenable from a physical point of view. As mentioned in sec. III A, a transgression form can be regarded as the difference of two CS forms plus a boundary term,

$$Q_{\boldsymbol{A}\leftarrow\bar{\boldsymbol{A}}}^{(2n+1)} = Q_{\boldsymbol{A}\leftarrow 0}^{(2n+1)} - Q_{\bar{\boldsymbol{A}}\leftarrow 0}^{(2n+1)} + \mathrm{d}Q_{\boldsymbol{A}\leftarrow 0\leftarrow\bar{\boldsymbol{A}}}^{(2n)}. \tag{76}$$

This means that a transgression action defined as $S_{\mathrm{T}}^{(2n+1)}\left[\boldsymbol{A}, \bar{\boldsymbol{A}}\right] = \int_M Q_{\boldsymbol{A}\leftarrow\bar{\boldsymbol{A}}}^{(2n+1)}$ describes the dynamics of two independent CS theories which interact only at the boundary of the space-time manifold M. However, as pointed out in Ref. [20], the kinetic term for $\bar{\boldsymbol{A}}$ has the 'wrong' sign, and therefore, its associated propagator will be ill-defined.

An interesting solution to this problem [20] consists in defining two different, but cobordant (i.e., sharing the same boundary) manifolds, M and $\bar{M}$. The connection $\bar{\boldsymbol{A}}$ is imposed to vanish on M, while $\boldsymbol{A}$ is imposed to vanish on $\bar{M}$; the sign difference is understood as due to the opposed orientations of M and $\bar{M}$ (see Fig. 2).

As a consequence, the action can no longer be considered to be the integral of the transgression form on a single manifold with boundary.

Here we shall suggest a slightly different point of view. The action will be given by the integral

$$S_{\mathrm{T}}^{(2n+1)}\left[\boldsymbol{A}, \bar{\boldsymbol{A}}\right] = \int_M Q_{\boldsymbol{A}\leftarrow\bar{\boldsymbol{A}}}^{(2n+1)}, \tag{77}$$

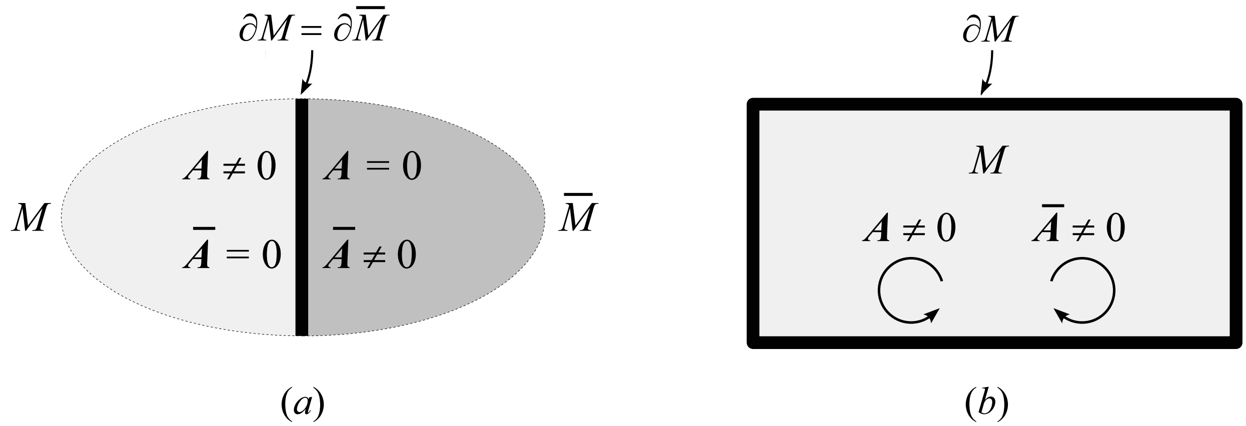

FIG. 2: (a) One solution to the sign problem in eq. (76) comes from considering two cobordant manifolds M and $\bar{M}$. The connection $\bar{\boldsymbol{A}}$ is imposed to vanish on M, while $\boldsymbol{A}$ is imposed to vanish on $\bar{M}$; the sign difference is understood as due to the opposed orientations of M and $\bar{M}$.
(b) An alternative solution to the sign problem in eq. (76) involves associating each connection to one of the two possible orientations in M. The sign difference is understood as coming from integrating on M with the opposite orientation.

and no condition will be *a priori* imposed on the connections. In order to solve the sign problem, the connections $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$ will be associated to both possible orientations of M, as depicted in Fig. 2.

In this way, the integral $-\int_M Q^{(2n+1)}_{\bar{\boldsymbol{A}}\leftarrow 0}$ corresponds to integrating $Q^{(2n+1)}_{\bar{\boldsymbol{A}}\leftarrow 0}$ on M but with the *opposite* orientation, i.e.

$$-\int_M Q^{(2n+1)}_{\bar{\boldsymbol{A}}\leftarrow 0} = \int_{-M} Q^{(2n+1)}_{\bar{\boldsymbol{A}}\leftarrow 0}. \tag{78}$$

and the action reads

$$\begin{aligned} S^{(2n+1)}_{\mathrm{T}}\left[\boldsymbol{A},\bar{\boldsymbol{A}}\right] &= \int_M Q^{(2n+1)}_{\boldsymbol{A}\leftarrow\bar{\boldsymbol{A}}} \\ &= \int_M Q^{(2n+1)}_{\boldsymbol{A}\leftarrow 0} + \int_{-M} Q^{(2n+1)}_{\bar{\boldsymbol{A}}\leftarrow 0} + \int_{\partial M} Q^{(2n)}_{\boldsymbol{A}\leftarrow 0\leftarrow\bar{\boldsymbol{A}}}. \end{aligned} \tag{79}$$

Roughly speaking, in this approach there are still two independent CS theories, each ‘living on one side’ of M and interacting only at the boundary.

On the other hand, the orientation of M is matter of choice; there is no such thing as a ‘right side’ of M. As a consequence, the action has to have the same form in the connections $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$, and the boundary conditions should be the same for both connections. This can be accomplished in a very natural way (see sec. IV).

C. Subspace Separation Method

In a previous work [18, 21], a subspace separation method for transgression lagrangians was sketched. The goal of the method is to write the lagrangian $L_{\mathrm{T}}^{(2n+1)}$ in a way that faithfully reflects the subspace structure of the algebra. The method is based on the iterative use of the 'Triangle Equation' [cf. eq. (73)]

$$Q_{\boldsymbol{A}\leftarrow\bar{\boldsymbol{A}}}^{(2n+1)} = Q_{\boldsymbol{A}\leftarrow\tilde{\boldsymbol{A}}}^{(2n+1)} + Q_{\tilde{\boldsymbol{A}}\leftarrow\bar{\boldsymbol{A}}}^{(2n+1)} + \mathrm{d}Q_{\boldsymbol{A}\leftarrow\tilde{\boldsymbol{A}}\leftarrow\bar{\boldsymbol{A}}}^{(2n)}. \tag{80}$$

Eq. (80) expresses a transgression form $Q_{\boldsymbol{A}\leftarrow\bar{\boldsymbol{A}}}^{(2n+1)}$ as the sum of two transgression forms depending on an arbitrary one-form $\tilde{\boldsymbol{A}}$ plus a total derivative. This last term has the form

$$Q_{\boldsymbol{A}\leftarrow\tilde{\boldsymbol{A}}\leftarrow\bar{\boldsymbol{A}}}^{(2n)} \equiv n\left(n+1\right)\int_0^1 dt \int_0^t ds \left\langle \left(\boldsymbol{A}-\tilde{\boldsymbol{A}}\right)\left(\tilde{\boldsymbol{A}}-\bar{\boldsymbol{A}}\right)\boldsymbol{F}_{st}^{n-1}\right\rangle, \tag{81}$$

where

$$\boldsymbol{A}_{st} = \bar{\boldsymbol{A}} + s\left(\boldsymbol{A}-\tilde{\boldsymbol{A}}\right) + t\left(\tilde{\boldsymbol{A}}-\bar{\boldsymbol{A}}\right), \tag{82}$$

$$\boldsymbol{F}_{st} = \mathrm{d}\boldsymbol{A}_{st} + \boldsymbol{A}_{st}^2. \tag{83}$$

A first splitting of the lagrangian (69) is achieved by introducing the intermediate connection $\tilde{\boldsymbol{A}} = \bar{\boldsymbol{\omega}}$,

$$L\left(\boldsymbol{A},\bar{\boldsymbol{A}}\right) = Q_{\boldsymbol{A}\leftarrow\bar{\boldsymbol{\omega}}}^{(11)} + Q_{\bar{\boldsymbol{\omega}}\leftarrow\bar{\boldsymbol{A}}}^{(11)} + \mathrm{d}Q_{\boldsymbol{A}\leftarrow\bar{\boldsymbol{\omega}}\leftarrow\bar{\boldsymbol{A}}}^{(10)}, \tag{84}$$

and a second one by separating $Q_{\boldsymbol{A}\leftarrow\bar{\boldsymbol{\omega}}}^{(11)}$ through $\boldsymbol{\omega}$:

$$Q_{\boldsymbol{A}\leftarrow\bar{\boldsymbol{\omega}}}^{(11)} = Q_{\boldsymbol{A}\leftarrow\boldsymbol{\omega}}^{(11)} + Q_{\boldsymbol{\omega}\leftarrow\bar{\boldsymbol{\omega}}}^{(11)} + \mathrm{d}Q_{\boldsymbol{A}\leftarrow\boldsymbol{\omega}\leftarrow\bar{\boldsymbol{\omega}}}^{(10)}. \tag{85}$$

After these two splittings, the lagrangian (69) reads

$$L\left(\boldsymbol{A},\bar{\boldsymbol{A}}\right) = Q_{\boldsymbol{A}\leftarrow\boldsymbol{\omega}}^{(11)} - Q_{\bar{\boldsymbol{A}}\leftarrow\bar{\boldsymbol{\omega}}}^{(11)} + Q_{\boldsymbol{\omega}\leftarrow\bar{\boldsymbol{\omega}}}^{(11)} + \mathrm{d}B^{(10)}, \tag{86}$$

with

$$B^{(10)} = Q_{\boldsymbol{A}\leftarrow\boldsymbol{\omega}\leftarrow\bar{\boldsymbol{\omega}}}^{(10)} + Q_{\boldsymbol{A}\leftarrow\bar{\boldsymbol{\omega}}\leftarrow\bar{\boldsymbol{A}}}^{(10)}. \tag{87}$$

The first two terms in (86) are identical (with the obvious replacements), and we shall mainly concentrate on analyzing them. The third term will be shown to be unrelated to the two former; in particular, it can be made to vanish without affecting the rest. The boundary term (87) can be written in a more explicit way by going back to eq. (81) and replacing the relevant connections and curvatures. The result is however not particularly illuminating and, as its explicit form is not needed in order to write boundary conditions, we shall not elaborate any longer on it.

D. M-Algebra Lagrangian

Let us examine the transgression $Q^{(11)}_{\boldsymbol{A}\leftarrow\boldsymbol{\omega}}$. The subspace separation method can be used again in order to write down a closed expression for it. To this end we introduce the following set of intermediate connections:

$$\boldsymbol{A}_0 = \boldsymbol{\omega}, \tag{88}$$

$$\boldsymbol{A}_1 = \boldsymbol{\omega} + \boldsymbol{e}, \tag{89}$$

$$\boldsymbol{A}_2 = \boldsymbol{\omega} + \boldsymbol{e} + \boldsymbol{b}_2, \tag{90}$$

$$\boldsymbol{A}_3 = \boldsymbol{\omega} + \boldsymbol{e} + \boldsymbol{b}_2 + \boldsymbol{b}_5, \tag{91}$$

$$\boldsymbol{A}_4 = \boldsymbol{\omega} + \boldsymbol{e} + \boldsymbol{b}_2 + \boldsymbol{b}_5 + \bar{\boldsymbol{\psi}}. \tag{92}$$

The Triangle Equation (80) allows us to split the transgression $Q^{(11)}_{\boldsymbol{A}_4\leftarrow\boldsymbol{A}_0}$ following the pattern

$$Q^{(11)}_{\boldsymbol{A}_4\leftarrow\boldsymbol{A}_0} = Q^{(11)}_{\boldsymbol{A}_4\leftarrow\boldsymbol{A}_3} + Q^{(11)}_{\boldsymbol{A}_3\leftarrow\boldsymbol{A}_0} + \mathrm{d}Q^{(10)}_{\boldsymbol{A}_4\leftarrow\boldsymbol{A}_3\leftarrow\boldsymbol{A}_0}, \tag{93}$$

$$Q^{(11)}_{\boldsymbol{A}_3\leftarrow\boldsymbol{A}_0} = Q^{(11)}_{\boldsymbol{A}_3\leftarrow\boldsymbol{A}_2} + Q^{(11)}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_0} + \mathrm{d}Q^{(10)}_{\boldsymbol{A}_3\leftarrow\boldsymbol{A}_2\leftarrow\boldsymbol{A}_0}, \tag{94}$$

$$Q^{(11)}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_0} = Q^{(11)}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_1} + Q^{(11)}_{\boldsymbol{A}_1\leftarrow\boldsymbol{A}_0} + \mathrm{d}Q^{(10)}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_1\leftarrow\boldsymbol{A}_0}. \tag{95}$$

Proceeding along these lines one arrives at the lagrangian

$$Q^{(11)}_{\boldsymbol{A}_4\leftarrow\boldsymbol{A}_0} = 6\left[H_a e^a + \frac{1}{2}H_{ab}b_2^{ab} + \frac{1}{5!}H_{abcde}b_5^{abcde} - \frac{5}{2}\bar{\psi}\mathcal{R}\mathrm{D}_\omega\psi\right]. \tag{96}$$

All three boundary terms that should in principle appear in (96) cancel due to the very particular properties of the invariant tensor chosen [cf. eqs. (45)–(49)].

The tensors H_a, H_{ab}, H_{abcde} and $\mathcal{R}$ are defined as

$$H_a \equiv \left\langle \boldsymbol{R}^5 \boldsymbol{P}_a \right\rangle_{\mathrm{M}}, \tag{97}$$

$$H_{ab} \equiv \left\langle \boldsymbol{R}^5 \boldsymbol{Z}_{ab} \right\rangle_{\mathrm{M}}, \tag{98}$$

$$H_{abcde} \equiv \left\langle \boldsymbol{R}^5 \boldsymbol{Z}_{abcde} \right\rangle_{\mathrm{M}}, \tag{99}$$

$$\mathcal{R}^\alpha_{\ \beta} \equiv \left\langle \boldsymbol{Q}^\alpha \boldsymbol{R}^4 \bar{\boldsymbol{Q}}_\beta \right\rangle_{\mathrm{M}}, \tag{100}$$

Explicitly using the invariant tensor (50)–(53) one finds

$$H_a = \frac{\alpha_2}{64} R_a^{(5)}, \tag{101}$$

$$H_{ab} = \alpha_2 \left[\frac{5}{2} \left(R^4 - \frac{3}{4} R^2 R^2 \right) R_{ab} + 5 R^2 R_{ab}^3 - 8 R_{ab}^5 \right], \tag{102}$$

$$H_{abcde} = -\frac{5}{16} \alpha_2 \left[5 R_{[ab} R_{cde]}^{(4)} - 40 R^f{}_{[a} R^g{}_b R_{cde]fg}^{(3)} - R^2 R_{abcde}^{(3)} + 4 R_{abcdefg}^{(2)} \left(R^3 \right)^{fg} \right], \tag{103}$$

$$\mathcal{R} = -\frac{\alpha_2}{40} \left\{ \left(R^4 - \frac{3}{4} R^2 R^2 \right) \mathbb{1} + \frac{1}{96} R_{abc}^{(4)} \Gamma^{abc} - \frac{3}{4} \left[R^2 R^{ab} - \frac{8}{3} \left(R^3 \right)^{ab} \right] R^{cd} \Gamma_{abcd} \right\}. \tag{104}$$

Here we have used the shortcuts

$$R^{2n} = R^{a_1}{}_{a_2} \cdots R^{a_{2n}}{}_{a_1}, \tag{105}$$

$$R_{ab}^{2n+1} = A_{ac_1} A^{c_1}{}_{c_2} \cdots A^{c_{2n}}{}_b, \tag{106}$$

$$R_{a_1 \cdots a_{11-2n}}^{(n)} = \varepsilon_{a_1 \cdots a_{11-2n} b_1 \cdots b_{2n}} R^{b_1 b_2} \cdots R^{b_{2n-1} b_{2n}}. \tag{107}$$

On section IV we shall comment on the dynamics produced by this lagrangian; here we may already note that no derivatives of e^a, b_2^{ab} or b_5^{abcde} appear. This can be traced back to the particular form of the invariant tensor (45)–(49), which contains no nonzero components of the form $\langle \boldsymbol{J}^3 \boldsymbol{P} \boldsymbol{Z}_2 \rangle$, etc.

The last contribution to the lagrangian (86) comes from the $Q_{\boldsymbol{\omega} \leftarrow \bar{\boldsymbol{\omega}}}$ term. Taking into account the definition of a transgression form and the form of the invariant tensor, it is straightforward to write down the expression

$$Q_{\boldsymbol{\omega} \leftarrow \bar{\boldsymbol{\omega}}}^{(11)} = 3 \int_0^1 \mathrm{d}t \; \theta^{ab} L_{ab}(t), \tag{108}$$

where

$$L_{ab}(t) = \left\langle \boldsymbol{R}_t^5 \boldsymbol{J}_{ab} \right\rangle_{\mathrm{M}} \tag{109}$$

and

$$\boldsymbol{R}_t = \frac{1}{2} \left[R_t \right]^{ab} \boldsymbol{J}_{ab}, \tag{110}$$

$$\left[R_t \right]^{ab} = \bar{R}^{ab} + t \mathrm{D}_{\bar{\boldsymbol{\omega}}} \theta^{ab} + t^2 \theta^a{}_c \theta^{cb}. \tag{111}$$

An explicit version for $L_{ab}(t)$ reads

$$L_{ab}(t) = \alpha_0 \left[\frac{5}{2} \left(R_t^4 - \frac{3}{4} R_t^2 R_t^2 \right) \left[R_t \right]_{ab} + 5 R_t^2 \left[R_t \right]_{ab}^3 - 8 \left[R_t \right]_{ab}^5 \right]. \tag{112}$$

A few comments are in order. As seen in (112), $Q^{(11)}_{\boldsymbol{\omega}\leftarrow\bar{\boldsymbol{\omega}}}$ is proportional to α_0, as opposed to all other terms, which are proportional to α_2. This is a direct consequence of the choice of invariant tensor. Being the only piece in the lagrangian unrelated to α_2, it can be removed by simply picking $\alpha_0 = 0$. This independence also means that $Q^{(11)}_{\boldsymbol{\omega}\leftarrow\bar{\boldsymbol{\omega}}}$ is by itself invariant under the M Algebra. This is related to the fact that this term corresponds to the only surviving component when the supertrace is used to construct the invariant tensor.

Because of its form, $Q^{(11)}_{\boldsymbol{\omega}\leftarrow\bar{\boldsymbol{\omega}}}$ apparently contains a bulk interaction of the $\boldsymbol{\omega}$ and $\bar{\boldsymbol{\omega}}$ fields. This is no more than an illusion; in order to realize this, it suffices to use the 'Triangle Equation' with the middle connection set to zero,

$$Q^{(11)}_{\boldsymbol{\omega}\leftarrow\bar{\boldsymbol{\omega}}} = Q^{(11)}_{\boldsymbol{\omega}\leftarrow 0} - Q^{(11)}_{\bar{\boldsymbol{\omega}}\leftarrow 0} + \mathrm{d}Q^{(10)}_{\boldsymbol{\omega}\leftarrow 0\leftarrow\bar{\boldsymbol{\omega}}}. \tag{113}$$

Here $Q^{(11)}_{\boldsymbol{\omega}\leftarrow 0}$ and $Q^{(11)}_{\bar{\boldsymbol{\omega}}\leftarrow 0}$ correspond to two independient CS exotic-gravity Lagrangians and $Q^{(10)}_{\boldsymbol{\omega}\leftarrow 0\leftarrow\bar{\boldsymbol{\omega}}}$ corresponds to the boundary piece relating them.

E. Relaxing Coupling Constants

All results so far have been obtained from the invariant tensor given in eqs. (50)–(53). This in turn was derived from the supersymmetrized supertrace of the product of six supermatrices representing as many M-Algebra generators. In particular, we have used 32×32 Dirac Matrices in $d = 11$ to represent the bosonic sector, so that the bosonic components of the invariant tensor correspond to their symmetrized trace [23, 24].

Different invariant tensors may be obtained by considering symmetrized products of traces, as in $\langle \boldsymbol{F}^p \rangle \langle \boldsymbol{F}^{n-p} \rangle$. To exhaust all possibilities one must consider the partitions of six (which is the order of the desired invariant tensor). A moment's thought shows that, apart from the already considered $6 = 6$ partition, only the $6 = 4 + 2$ and $6 = 2 + 2 + 2$ cases contribute, as all others identically vanish. We are thus led to consider the following linear combination:

$$\langle \cdots \rangle_{\mathrm{M}} = \langle \cdots \rangle_{6=6} + \beta_{4+2} \langle \cdots \rangle_{6=4+2} + \beta_{2+2+2} \langle \cdots \rangle_{6=2+2+2} \,. \tag{114}$$

(The coefficient in front of $\langle \cdots \rangle_{6=6}$ can be normalized to unity without any loss of generality).

The amazing result of performing this exercise is that no new terms appear in the invariant tensor (114); rather, the original rigid structure found in (50)–(53) is relaxed into one which takes into account the new coupling constants β_{4+2} and β_{2+2+2}. Turning these constants on and off one finds that there are several distinct sectors which are by themselves invariant, so that it is perfectly sensible to associate them with different couplings.

The net effect on the lagrangian (96) concerns only the explicit expressions for the tensors defined in (97)–(100); the new versions read

$$H_a = \frac{\alpha_2}{64} R^{(5)}_a, \tag{115}$$

$$H_{ab} = \alpha_2 \left[\frac{5}{2}\left(\kappa_{15} R^4 - \frac{3}{4}\gamma_5 R^2 R^2\right) R_{ab} + 5\kappa_{15} R^2 R^3_{ab} - 8 R^5_{ab}\right], \tag{116}$$

$$H_{abcde} = -\frac{5}{16}\alpha_2 \left[5 R_{[ab} R^{(4)}_{cde]} + 40 R^f_{\ [a} R^g_{\ b} R^{(3)}_{cde]fg} - \kappa_{15} R^2 R^{(3)}_{abcde} + 4 R^{(2)}_{abcdefg}\left(R^3\right)^{fg}\right], \tag{117}$$

$$\begin{aligned}\mathcal{R} = & -\frac{\alpha_2}{40}\left\{\left[\kappa_3 R^4 - \frac{3}{4}\left(5\gamma_9 - 4\right) R^2 R^2\right] \mathbb{1} + \frac{1}{96} R^{(4)}_{abc}\Gamma^{abc} + \right. \\ & \left. -\frac{3}{4}\left[\kappa_9 R^2 R^{ab} - \frac{8}{3}\left(R^3\right)^{ab}\right] R^{cd}\Gamma_{abcd}\right\}.\end{aligned} \tag{118}$$

The constants κ_n and γ_n are not, as it may seem, an infinite tower of arbitrary coupling constants, but are rather tightly constrained by the relations

$$\kappa_m = 1 + \frac{n}{m}\left(\kappa_n - 1\right), \tag{119}$$

$$\gamma_m = \gamma_n + \left(\frac{n}{m} - 1\right)\left(\kappa_n - 1\right). \tag{120}$$

These two sets of constants replace the above β_{4+2} and β_{2+2+2}; once a representative from every one of them has been chosen, the rest is univocally determined by (119)–(120). In other words, fixing one particular κ_n sets the values of all others. Once all κ_n are fixed, choosing one γ_n ties together all the γ's.

The original coupling constants β_{4+2} and β_{2+2+2} can be expressed in terms of the new κ_n and γ_n as [29]

$$\beta_{4+2} = \frac{1}{\mathrm{Tr}\left(\mathbb{1}\right)} n\left(\kappa_n - 1\right), \tag{121}$$

$$\beta_{2+2+2} = \frac{15}{\left[\mathrm{Tr}\left(\mathbb{1}\right)\right]^2}\left(\gamma_n - \kappa_n\right). \tag{122}$$

It is also worth to notice that

$$\beta_{4+2} = 0 \qquad \Leftrightarrow \qquad \kappa_n = 1, \tag{123}$$

$$\beta_{2+2+2} = 0 \qquad \Leftrightarrow \qquad \gamma_n = \kappa_n. \tag{124}$$

IV. DYNAMICS

A. Field Equations and Four-Dimensional Dynamics

The field equations for $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$ are completely analogous, and therefore in this section they will be presented only for $\boldsymbol{A}$. The general expression for the field equations can be read off directly from the variation (75), and has the form

$$\left\langle \boldsymbol{F}^5 \boldsymbol{T}_A \right\rangle_{\mathrm{M}} = 0, \tag{125}$$

where $\{\boldsymbol{T}_A, A = 1, \ldots, \dim(\mathfrak{g})\}$ is a basis for the algebra and $\boldsymbol{F}$ is the curvature.

The field equations obtained by varying e^a, b_2^{ab}, $b_5^{a_1 \cdots a_5}$ and ψ are given by

$$H_a = 0, \tag{126}$$

$$H_{ab} = 0, \tag{127}$$

$$H_{abcde} = 0, \tag{128}$$

$$\mathcal{R} \mathrm{D}_\omega \psi = 0, \tag{129}$$

where explicit expressions for H_a, H_{ab}, H_{abcde} and $\mathcal{R}$ can be found in (101)–(104) [but see also sec. III E and eqs. (115)–(118)]. The field equation obtained from varying ω^{ab} reads

$$\begin{aligned} & L_{ab} - 10 \left(\mathrm{D}_\omega \bar{\psi} \right) \mathcal{Z}_{ab} \left(\mathrm{D}_\omega \psi \right) + 5 H_{abc} \left(T^c + \frac{1}{16} \bar{\psi} \Gamma^c \psi \right) + \\ & + \frac{5}{2} H_{abcd} \left(\mathrm{D}_\omega b^{cd} - \frac{1}{16} \bar{\psi} \Gamma^{cd} \psi \right) + \frac{1}{24} H_{abc_1 \cdots c_5} \left(\mathrm{D}_\omega b^{c_1 \cdots c_5} + \frac{1}{16} \bar{\psi} \Gamma^{c_1 \cdots c_5} \psi \right) = 0, \end{aligned} \tag{130}$$

where we have defined

$$L_{ab} \equiv \left\langle \boldsymbol{R}^5 \boldsymbol{J}_{ab} \right\rangle_{\mathrm{M}}, \tag{131}$$

$$\left(\mathcal{Z}_{ab} \right)^\alpha_{\ \beta} \equiv \left\langle \boldsymbol{Q}^\alpha \boldsymbol{R}^3 \boldsymbol{J}_{ab} \bar{\boldsymbol{Q}}_\beta \right\rangle_{\mathrm{M}}, \tag{132}$$

$$H_{abc} \equiv \left\langle \boldsymbol{R}^4 \boldsymbol{J}_{ab} \boldsymbol{P}_c \right\rangle_{\mathrm{M}}, \tag{133}$$

$$H_{abcd} \equiv \left\langle \boldsymbol{R}^4 \boldsymbol{J}_{ab} \boldsymbol{Z}_{cd} \right\rangle_{\mathrm{M}}, \tag{134}$$

$$H_{abcdefg} \equiv \left\langle \boldsymbol{R}^4 \boldsymbol{J}_{ab} \boldsymbol{Z}_{cdefg} \right\rangle_{\mathrm{M}}. \tag{135}$$

Explicit versions for these quantities are found using the invariant tensor (50)–(53):

$$L_{ab} = \alpha_0 \left[\frac{5}{2} \left(R^4 - \frac{3}{4} R^2 R^2 \right) R_{ab} + 5 R^2 R^3_{ab} - 8 R^5_{ab} \right], \tag{136}$$

$$
\begin{aligned}
\mathcal{Z}_{ab} \;=\; & \frac{\alpha_2}{40}\left(2\left[R^3_{ab}-\frac{3}{4}R^2R_{ab}\right]\mathbb{1}-\frac{1}{48}R^{(3)}_{abcde}\Gamma^{cde}+\right. \\
& -\frac{3}{4}\left(R_{ab}R^{cd}-\frac{1}{2}R^2\delta^{cd}_{ab}\right)R^{ef}\Gamma_{cdef}+ \\
& \left.-\left[\delta^{cg}_{ab}R_{gh}R^{hd}R^{ef}-R^c{}_aR^d{}_bR^{ef}+\frac{1}{2}\delta^{ef}_{ab}\left(R^3\right)^{cd}\right]\Gamma_{cdef}\right),
\end{aligned}
\tag{137}
$$

$$
\begin{aligned}
H_{abcd} \;=\; & \alpha_2\delta^{ef}_{ab}\delta^{gh}_{cd}\left[\frac{3}{4}R^2R_{ef}R_{gh}-R^3_{ef}R_{gh}-R_{ef}R^3_{gh}+\right. \\
& -\frac{4}{5}\left(R_{eh}R^3_{fg}+R^3_{eh}R_{fg}-R^2_{eh}R^2_{fg}\right)+\frac{1}{2}R^2R_{eh}R_{fg}+ \\
& \left.+\frac{1}{8}\eta_{[ef][gh]}\left(R^4-\frac{3}{4}\gamma_5R^2R^2\right)-\eta_{fg}\left(R^2R^2_{eh}-\frac{8}{5}R^4_{eh}\right)\right],
\end{aligned}
\tag{138}
$$

$$
\begin{aligned}
H_{abc_1\cdots c_5} \;=\; & \frac{\alpha_2}{80}\delta^{cdefg}_{c_1\cdots c_5}\left(-\frac{5}{3}R^{(3)}_{abcde}R_{fg}+10R^{(2)}_{abcdepq}R^p{}_fR^q{}_g+\right. \\
& -\frac{1}{6}R_{ab}R^{(3)}_{cdefg}+\frac{1}{4}R^2R^{(2)}_{abcdefg}-\frac{2}{3}R^{(1)}_{abcdefgpq}\left(R^3\right)^{pq}+ \\
& +\frac{1}{3}R^p{}_aR^q{}_bR^{(2)}_{cdefgpq}-\frac{1}{3}R^q{}_aR^{(2)}_{bcdefgp}R^p{}_q+\frac{1}{3}R^q{}_bR^{(2)}_{acdefgp}R^p{}_q+ \\
& \left.-\frac{10}{3}\eta_{ga}R^{(3)}_{bcdep}R^p{}_f+\frac{10}{3}\eta_{gb}R^{(3)}_{acdep}R^p{}_f-\frac{5}{24}\eta_{[ab][cd]}R^{(4)}_{efg}\right).
\end{aligned}
\tag{139}
$$

They satisfy the relationships

$$H_c = \frac{1}{2}R^{ab}H_{abc}, \tag{140}$$

$$H_{cd} = \frac{1}{2}R^{ab}H_{abcd}, \tag{141}$$

$$H_{cdefg} = \frac{1}{2}R^{ab}H_{abcdefg}, \tag{142}$$

$$\mathcal{R}^\alpha_\beta = \frac{1}{2}R^{ab}\left(\mathcal{Z}_{ab}\right)^\alpha_{\ \beta}. \tag{143}$$

As a first application of the field equations, we discuss the problem of the vacuum.

Finding a 'true vacuum' for a transgression theory is an interesting, non-trivial problem. The natural candidate for the vacuum is $\boldsymbol{F}=0$; this configuration satisfies the field equations, is stable, with zero charges and fully gauge-invariant. As discussed in Refs. [1, 2, 3, 13, 15], when $d=2n+1\geq 5$ there is a big problem: perturbations do not propagate around this background, as can be directly seen from the field equations,

$$\left\langle \boldsymbol{F}^n\boldsymbol{T}_A\right\rangle = 0. \tag{144}$$

This means that there are no local propagating degrees of freedom.

Several solutions have been offered for this problem. In Ref. [2], matter interaction in terms of Wilson lines was added in order to have local propagating degrees of freedom around the $\boldsymbol{F} = 0$ background. An alternative and particularly elegant solution was proposed in [13, 15]: no extra term is added, but the vacuum is allowed to have $\boldsymbol{F} \neq 0$, as long as eq. (144) is satisfied as a simple zero, allowing for perturbations to propagate. An amazing consequence of this last approach is that the requirement of having propagating degrees of freedom picks up by itself a configuration with a four-dimensional domain-wall universe as solution.

Here we analyze how this model fits within the present theory, which has a different lagrangian as the one in [13, 15]. We also explore the consequences of allowing the eleven-dimensional torsion to be nonzero.

Consider the geometrical ansatz $M = X_{d+1} \times S^{10-d}$, where X_{d+1} is the warped product of a d-dimensional domain wall M_d and $\mathbb{R}$, with S^{10-d} corresponding to a non-flat $(10-d)$-dimensional manifold with constant curvature and zero torsion. The metric for this case corresponds to eq. (4.3) from Ref. [15],

$$\mathrm{d}s^2 = e^{-2\xi|z|} \left(\mathrm{d}z^2 + \tilde{g}^{(d)}_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta \right) + \gamma^{(10-d)}_{\kappa\lambda} \mathrm{d}y^\kappa \mathrm{d}y^\lambda. \tag{145}$$

For this particular section only, we use the index Z for the tangent space of $\mathbb{R}$, $a, b, c, \ldots$ for the tangent space of M_d and $i, j, k, \ldots$ for the tangent space of S^{10-d}. The components of the curvature and torsion read

$$R^{ab} = \tilde{R}^{ab} - \xi^2 \tilde{e}^a \tilde{e}^b + 2\xi\theta(z) \left(\tilde{e}^a \kappa^b - \tilde{e}^b \kappa^a \right) - \kappa^a \kappa^b, \tag{146}$$

$$R^{aZ} = -2e^{\xi|z|} \xi \delta(z) E^Z \tilde{e}^a - 2\xi\theta(z) \tilde{T}^a + \mathrm{D}_{\tilde{\omega}} \kappa^a, \tag{147}$$

$$T^a = \kappa^a E^Z + e^{-\xi|z|} \tilde{T}^a, \tag{148}$$

$$T^Z = -e^{-\xi|z|} \kappa^a \tilde{e}_a. \tag{149}$$

Here $\tilde{R}^{ab}$ and $\tilde{T}^a$ correspond to the M_d curvature and torsion, and κ^a corresponds to the k^{aZ} component of the eleven-dimensional contorsion. The Heaviside function and Dirac's delta are denoted as usual by $\theta(z)$ and $\delta(z)$ respectively. When the equation of motion (126) is taken into account, it is possible to prove, following the same arguments as in Refs. [13, 15], that the only way of having propagating degrees of freedom is imposing $d = 4$.

In this case ($d=4$), the field equation (126) splits into

$$\xi e^{\xi|z|}\delta(z)\,E^{Z}\varepsilon_{abcd}\left(\tilde{R}^{ab}-\xi^{2}\tilde{e}^{a}\tilde{e}^{b}\right)\tilde{e}^{c} \;=\; \mathcal{T}_{d}, \tag{150}$$

$$\varepsilon_{abcd}\left(\tilde{R}^{ab}-\xi^{2}\tilde{e}^{a}\tilde{e}^{b}\right)\left(\tilde{R}^{cd}-\xi^{2}\tilde{e}^{c}\tilde{e}^{d}\right) \;=\; \mathcal{T}, \tag{151}$$

with

$$\begin{aligned}\mathcal{T}_{d} \;=\;& 2E^{Z}e^{\xi|z|}\xi\delta(z)\,\varepsilon_{abcd}\left(\frac{1}{2}\kappa^{a}\kappa^{b}-\xi\theta(z)\left(\tilde{e}^{a}\kappa^{b}-\tilde{e}^{b}\kappa^{a}\right)\right)\tilde{e}^{c}+\\ &+\varepsilon_{abcd}\left[\tilde{R}^{ab}-\xi^{2}\tilde{e}^{a}\tilde{e}^{b}+2\xi\theta(z)\left(\tilde{e}^{a}\kappa^{b}-\tilde{e}^{b}\kappa^{a}\right)-\kappa^{a}\kappa^{b}\right]\left(\frac{1}{2}\mathrm{D}_{\tilde{\omega}}\kappa^{c}-\xi\theta(z)\,\tilde{T}^{c}\right),\end{aligned} \tag{152}$$

$$\begin{aligned}\mathcal{T} \;=\;& -4\varepsilon_{abcd}\left(\tilde{R}^{ab}-\xi^{2}\tilde{e}^{a}\tilde{e}^{b}+\xi\theta(z)\left(\tilde{e}^{a}\kappa^{b}-\tilde{e}^{b}\kappa^{a}\right)-\frac{1}{2}\kappa^{a}\kappa^{b}\right)\times\\ &\times\left(\xi\theta(z)\left(\tilde{e}^{c}\kappa^{d}-\tilde{e}^{d}\kappa^{c}\right)-\frac{1}{2}\kappa^{c}\kappa^{d}\right).\end{aligned} \tag{153}$$

The first of these equations of motion corresponds to the Einstein equations, with support limited to M_4. The right-hand side of this equation contains coupling among gravity and torsion. Even setting the four-dimensional torsion $\tilde{T}^a$ equal to zero, its remaining components κ^a camouflage as some sort of matter as seen from a four-dimensional point of view [see eq. (148)]. The second equation imposes extra relationships between the four-dimensional geometry and κ^a. Furthermore, the equations of motion (127) and (128) impose even more constraints on the geometry.

In contrast, the equations of motion (129) and (130) relate the four-dimensional geometry with κ^a, the fermions and the central charges.

In this way, it seems that there are too many constraints on the four-dimensional geometry as to reproduce four-dimensional General Relativity (for an analysis of a similar situation which arises in five dimensions, see Ref. [25]).

There are several ways in which one could deal with this problem; as we will discuss in the conclusions, the excess of constraints is strongly related to the semigroup choice made in order to construct the M Algebra and also to the 0_S-forcing. When other semigroups are chosen, different algebras can arise which reproduce several features of the M Algebra without having its 'dynamical rigidity' [10].

B. Symmetric Boundary Conditions

As explained in sec. III B, the boundary conditions should be completely equivalent for both connections. In this sense, the boundary conditions deduced from eq. (75) look a bit 'asymmetrical' in the rôles of $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$,

$$n\left(n+1\right)k\int_{0}^{1}\mathrm{d}t\left\langle\left(\delta\bar{\boldsymbol{A}}+t\delta\boldsymbol{\Theta}\right)\boldsymbol{\Theta}\boldsymbol{F}_{t}^{n-1}\right\rangle=0. \tag{154}$$

Here, it seems natural to solve the boundary conditions imposing

$$\left\langle\delta\bar{\boldsymbol{A}}\boldsymbol{\Theta}\boldsymbol{T}_{A_1}\cdots\boldsymbol{T}_{A_{n-1}}\right\rangle=0, \tag{155}$$

$$\left\langle\delta\boldsymbol{\Theta}\boldsymbol{\Theta}\boldsymbol{T}_{A_1}\cdots\boldsymbol{T}_{A_{n-1}}\right\rangle=0, \tag{156}$$

and therefore,

$$\delta\bar{\boldsymbol{A}}=0, \tag{157}$$

$$\delta\boldsymbol{\Theta}=\delta\boldsymbol{A}, \tag{158}$$

$$\left\langle\delta\boldsymbol{\Theta}\boldsymbol{\Theta}\boldsymbol{T}_{A_1}\cdots\boldsymbol{T}_{A_{n-1}}\right\rangle=0. \tag{159}$$

Examples of this fixing of boundary conditions in the context of gravity are given in Refs. [18, 19, 20, 21].

On the other hand, this 'asymmetry' between $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$ is only virtual; it is enough to perform the change of variables $t\to1-t$ in eq. (154) to obtain the reciprocal boundary condition. Following this approach, it seems interesting to consider as a solution to both the original boundary condition (154) and its reciprocal the relations

$$\left\langle\left(\delta\bar{\boldsymbol{A}}+t\delta\boldsymbol{\Theta}\right)\boldsymbol{\Theta}\boldsymbol{T}_{A_1}\cdots\boldsymbol{T}_{A_{n-1}}\right\rangle=0, \tag{160}$$

$$\left\langle\left(\delta\boldsymbol{A}-t\delta\boldsymbol{\Theta}\right)\boldsymbol{\Theta}\boldsymbol{T}_{A_1}\cdots\boldsymbol{T}_{A_{n-1}}\right\rangle=0, \tag{161}$$

with t arbitrary. From them we arrive at explicitly 'symmetrical' boundary conditions,

$$\left\langle\left(\delta\bar{\boldsymbol{A}}+\delta\boldsymbol{A}\right)\boldsymbol{\Theta}\boldsymbol{T}_{A_1}\cdots\boldsymbol{T}_{A_{n-1}}\right\rangle=0, \tag{162}$$

$$\left\langle\delta\boldsymbol{\Theta}\boldsymbol{\Theta}\boldsymbol{T}_{A_1}\cdots\boldsymbol{T}_{A_{n-1}}\right\rangle=0. \tag{163}$$

These conditions can be solved by demanding

$$\delta\boldsymbol{A}=-\delta\bar{\boldsymbol{A}}=\frac{1}{2}\delta\boldsymbol{\Theta}, \tag{164}$$

$$\left\langle\delta\boldsymbol{\Theta}\boldsymbol{\Theta}\boldsymbol{T}_{A_1}\cdots\boldsymbol{T}_{A_{n-1}}\right\rangle=0. \tag{165}$$

While this is by no means a general solution, it provides a symmetrical way of fixing the boundary conditions for any transgression lagrangian.

In the M Algebra case, we have

$$\boldsymbol{A} = \boldsymbol{b}_2 + \boldsymbol{b}_5 + \boldsymbol{e} + \boldsymbol{\omega} + \bar{\boldsymbol{\psi}}, \tag{166}$$

$$\bar{\boldsymbol{A}} = \bar{\boldsymbol{b}}_2 + \bar{\boldsymbol{b}}_5 + \bar{\boldsymbol{e}} + \bar{\boldsymbol{\omega}} + \bar{\boldsymbol{\chi}}, \tag{167}$$

and

$$\boldsymbol{\theta} = \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}. \tag{168}$$

Due the peculiarities of the invariant tensor [cf. eq. (45)–(49)], eq. (165) can be solved by demanding

$$\left\langle \left(\delta\boldsymbol{\theta}\boldsymbol{e} + \delta\boldsymbol{e}\boldsymbol{\theta} \right) \boldsymbol{J}_{a_1 b_1} \cdots \boldsymbol{J}_{a_4 b_4} \right\rangle = 0, \tag{169}$$

$$\left\langle \left(\delta\boldsymbol{\theta}\bar{\boldsymbol{e}} + \delta\bar{\boldsymbol{e}}\boldsymbol{\theta} \right) \boldsymbol{J}_{a_1 b_1} \cdots \boldsymbol{J}_{a_4 b_4} \right\rangle = 0, \tag{170}$$

$$\left\langle \left(\delta\boldsymbol{\theta}\boldsymbol{b}_2 + \delta\boldsymbol{b}_2\boldsymbol{\theta} \right) \boldsymbol{J}_{a_1 b_1} \cdots \boldsymbol{J}_{a_4 b_4} \right\rangle = 0, \tag{171}$$

$$\left\langle \left(\delta\boldsymbol{\theta}\bar{\boldsymbol{b}}_2 + \delta\bar{\boldsymbol{b}}_2\boldsymbol{\theta} \right) \boldsymbol{J}_{a_1 b_1} \cdots \boldsymbol{J}_{a_4 b_4} \right\rangle = 0, \tag{172}$$

$$\left\langle \left(\delta\boldsymbol{\theta}\boldsymbol{b}_5 + \delta\boldsymbol{b}_5\boldsymbol{\theta} \right) \boldsymbol{J}_{a_1 b_1} \cdots \boldsymbol{J}_{a_4 b_4} \right\rangle = 0, \tag{173}$$

$$\left\langle \left(\delta\boldsymbol{\theta}\bar{\boldsymbol{b}}_5 + \delta\bar{\boldsymbol{b}}_5\boldsymbol{\theta} \right) \boldsymbol{J}_{a_1 b_1} \cdots \boldsymbol{J}_{a_4 b_4} \right\rangle = 0, \tag{174}$$

$$\left\langle \left(\delta\boldsymbol{\theta}\bar{\boldsymbol{\psi}} + \delta\bar{\boldsymbol{\psi}}\boldsymbol{\theta} \right) \boldsymbol{J}_{a_1 b_1} \boldsymbol{J}_{a_2 b_2} \boldsymbol{J}_{a_3 b_3} \bar{\boldsymbol{Q}} \right\rangle = 0, \tag{175}$$

$$\left\langle \left(\delta\boldsymbol{\theta}\bar{\boldsymbol{\chi}} + \delta\bar{\boldsymbol{\chi}}\boldsymbol{\theta} \right) \boldsymbol{J}_{a_1 b_1} \boldsymbol{J}_{a_2 b_2} \boldsymbol{J}_{a_3 b_3} \bar{\boldsymbol{Q}} \right\rangle = 0, \tag{176}$$

and

$$\left\langle \delta\boldsymbol{\theta}\boldsymbol{\theta} \boldsymbol{J}_{a_1 b_1} \cdots \boldsymbol{J}_{a_4 b_4} \right\rangle = 0, \tag{177}$$

$$\left\langle \delta\boldsymbol{\theta}\boldsymbol{\theta} \boldsymbol{J}_{a_1 b_1} \boldsymbol{J}_{a_2 b_2} \boldsymbol{Q} \bar{\boldsymbol{Q}} \right\rangle = 0, \tag{178}$$

$$\left\langle \left(\delta\bar{\boldsymbol{\psi}} + \delta\bar{\boldsymbol{\chi}} \right) \left(\boldsymbol{\psi} + \boldsymbol{\chi} \right) \boldsymbol{J}_{a_1 b_1} \cdots \boldsymbol{J}_{a_4 b_4} \right\rangle = 0. \tag{179}$$

Given that $\delta\boldsymbol{A} = -\delta\bar{\boldsymbol{A}}$, a natural requirement to solve the system (169)–(176) is

$$\left. \bar{\boldsymbol{e}} \right|_{\partial M} = - \left. \boldsymbol{e} \right|_{\partial M}, \tag{180}$$

$$\left. \bar{\boldsymbol{b}}_2 \right|_{\partial M} = - \left. \boldsymbol{b}_2 \right|_{\partial M}, \tag{181}$$

$$\left. \bar{\boldsymbol{b}}_5 \right|_{\partial M} = - \left. \boldsymbol{b}_5 \right|_{\partial M}, \tag{182}$$

$$\left. \boldsymbol{\chi} \right|_{\partial M} = - \left. \boldsymbol{\psi} \right|_{\partial M}. \tag{183}$$

It is noteworthy that the condition $\bar{\boldsymbol{e}}|_{\partial M} = -\,\boldsymbol{e}|_{\partial M}$ fits perfectly well with the idea of associating connection $\boldsymbol{A}$ with one orientation of M and $\bar{\boldsymbol{A}}$ with the opposite one (this is due to the fact that, in odd dimensions, the transformation $\boldsymbol{e} \to -\boldsymbol{e}$ changes the manifold orientation).

Using the condition $\delta\boldsymbol{A} = -\delta\bar{\boldsymbol{A}}$, eq. (179) is fulfilled automatically, and then, we are left with the boundary conditions

$$\begin{aligned}
\left\langle \left(\delta\boldsymbol{\theta}\boldsymbol{e} + \delta\boldsymbol{e}\boldsymbol{\theta}\right) \boldsymbol{J}_{a_1 b_1} \cdots \boldsymbol{J}_{a_4 b_4} \right\rangle &= 0, \\
\left\langle \left(\delta\boldsymbol{\theta}\boldsymbol{b}_2 + \delta\boldsymbol{b}_2\boldsymbol{\theta}\right) \boldsymbol{J}_{a_1 b_1} \cdots \boldsymbol{J}_{a_4 b_4} \right\rangle &= 0, \\
\left\langle \left(\delta\boldsymbol{\theta}\boldsymbol{b}_5 + \delta\boldsymbol{b}_5\boldsymbol{\theta}\right) \boldsymbol{J}_{a_1 b_1} \cdots \boldsymbol{J}_{a_4 b_4} \right\rangle &= 0, \\
\left\langle \left(\delta\boldsymbol{\theta}\bar{\boldsymbol{\psi}} + \delta\bar{\boldsymbol{\psi}}\boldsymbol{\theta}\right) \boldsymbol{J}_{a_1 b_1} \boldsymbol{J}_{a_2 b_2} \boldsymbol{J}_{a_3 b_3} \bar{\boldsymbol{Q}} \right\rangle &= 0, \\
\left\langle \delta\boldsymbol{\theta}\boldsymbol{\theta}\boldsymbol{J}_{a_1 b_1} \cdots \boldsymbol{J}_{a_4 b_4} \right\rangle &= 0, \\
\left\langle \delta\boldsymbol{\theta}\boldsymbol{\theta}\boldsymbol{J}_{a_1 b_1} \boldsymbol{J}_{a_2 b_2} \boldsymbol{Q}\bar{\boldsymbol{Q}} \right\rangle &= 0,
\end{aligned}$$

which are to be supplemented with $\delta\boldsymbol{A} = -\delta\bar{\boldsymbol{A}}$ and (180)–(183). It is interesting to observe that eq. (169) coincides with the boundary conditions for the case of pure-gravity (see Refs. [18, 19, 20, 21]) and that the extra conditions seem to be natural extension of it.

A very generic way of solving this set of equations is requiring $\delta\boldsymbol{A} = \delta\tau\boldsymbol{A}$, with $\delta\tau$ being an arbitrary infinitesimal parameter. This is highly ad-hoc, though, and it would seem more natural to take into account the explicit form of the invariant tensor in order to solve them.

V. SUMMARY AND CONCLUSIONS

The construction of a transgression gauge field theory for the M Algebra has been developed in a very straightforward way through the use of two sets of mathematical tools. The first of these sets was provided in Ref. [10], where the procedure of expansion is analyzed using abelian semigroups and non-trace invariant tensors for this kind of algebras are written. The problem of the invariant tensor is far from being a trivial one; as discussed in Ref. [10], the 0_S-forcing procedure which was necessary in order to construct the M Algebra from $\mathfrak{osp}\,(32|1)$ also renders the supertrace, usually used as invariant tensor, as almost useless. The other set of tools is related with properties of transgression forms, and specially with the subspaces separation method [18, 21], used in order to write down the Lagrangian in an explicit way. Without using this method, the explicit writing of the action becomes a long, highly non-trivial task, where integrations by parts must be performed several times in an 'artistic' way.

From a physical point of view, it is very compelling that, using the methods of 'dynamical dimensional reduction' introduced in [13, 15], something that looks like a 'frozen' version of four-dimensional Einstein–Hilbert gravity with positive cosmological constant is obtained by simply abandoning the prejudice that the vacuum should satisfy $\boldsymbol{F} = 0$. This dynamics 'freezing' is a consequence of the form of the constrained form of the invariant tensor: the M Algebra has *more* generators than $\mathfrak{osp}\,(32|1)$, but *less* non-vanishing components on the invariant tensor. For this reason, the equations of motion associated to the variations of e^a, b_2^{ab} and $b_5^{a_1\cdots a_5}$ becomes simply constraints on the gravitational sector. But the poor form of the invariant tensor is a direct consequence of the 0_S-forcing procedure. As shown in Theorem 4 from Ref. [10], an invariant tensor for a generic S-expanded algebra without 0_S-forcing has more non-vanishing components than its 0_S-forced counterpart and, in general, even more components than the invariant tensor of the original algebra.

The above considerations make it evident that it would be advisable to avoid the 0_S-forcing. The M Algebra arises as the 0_S-forcing of the resonant subalgebra given by eq. (24). This resonant subalgebra itself looks very much like the M Algebra, in the sense that it has the anticommutator

$$\left\{ \boldsymbol{Q}, \bar{\boldsymbol{Q}} \right\} = \frac{1}{8} \left(\Gamma^a \boldsymbol{P}_a - \frac{1}{2} \Gamma^{ab} \boldsymbol{Z}_{ab} + \frac{1}{5!} \Gamma^{a_1 \cdots a_5} \boldsymbol{Z}_{a_1 \cdots a_5} \right), \tag{184}$$

but it also has an $\mathfrak{osp}\,(32|1)$ subalgebra (spanned by $\lambda_3 \boldsymbol{J}_{ab}$, $\lambda_3 \boldsymbol{P}_a$, $\lambda_3 \boldsymbol{Z}_{a_1 \cdots a_5}$ and $\lambda_3 \boldsymbol{Q}$; let us remember that $\lambda_3 \lambda_3 = \lambda_3$). The 'central charges' are no longer abelian; rather, their commutators take values on the $\lambda_3 \otimes \mathfrak{osp}\,(32|1)$ sector. This algebra has a much bigger tensor than the 'normal' M Algebra (see Theorem 4 from Ref. [10]), and therefore, an 'unfrozen' dynamics which has good chances of reproducing four-dimensional Einstein–Hilbert Gravity.

A more elegant algebra choice is also shown in Ref. [10]. Replacing the M Algebra's semigroup $S_{\mathrm{E}}^{(2)}$ for the cyclic group $\mathbb{Z}_4$, a resonant subalgebra of $\mathbb{Z}_4 \otimes \mathfrak{osp}\,(32|1)$ is obtained. It has very interesting features, like having two fermionic charges, $\boldsymbol{Q}$ and $\boldsymbol{Q}'$ with an M Algebra-like anticommutator

$$\left\{ \boldsymbol{Q}', \bar{\boldsymbol{Q}}' \right\} = \left\{ \boldsymbol{Q}, \bar{\boldsymbol{Q}} \right\} = \frac{1}{8} \left(\Gamma^a \boldsymbol{P}_a - \frac{1}{2} \Gamma^{ab} \boldsymbol{Z}_{ab} + \frac{1}{5!} \Gamma^{a_1 \cdots a_5} \boldsymbol{Z}_{a_1 \cdots a_5} \right). \tag{185}$$

Two sets of AdS boost generators, $\boldsymbol{P}_a$ and $\boldsymbol{P}'_a$, and two (non-abelian) 'M5' generators, $\boldsymbol{Z}_{a_1 \cdots a_5}$ and $\boldsymbol{Z}'_{a_1 \cdots a_5}$, are also present. This doubling in several generators makes it specially suitable to construct a transgression gauge field theory. On the other hand, since $\mathbb{Z}_4$ is a discrete group, it does not have a zero element; therefore, it has from the outset very good chances of having unfrozen four-dimensional dynamics. Work regarding this issue will be presented elsewhere.

At this point, it is natural to ask ourselves what the relationship between this M Algebra or M Algebra-like transgression theories and M Theory could be. It has been proposed that some CS supergravity theories (see Refs. [1, 2, 3, 26]) in eleven dimensions could actually correspond to M Theory, but the potential relations to standard CJS supergravity and String theory remain unsettled. As already discussed, in order to solve these problems it might be wise to take into account the fact that the M Algebra is but one possible choice within a family of superalgebras. Other members of this family [obtained from $\mathfrak{osp}\,(32|1)$ using different abelian semigroups, for instance] might also play a rôle in finding a truly fundamental symmetry.

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[27] When the M Algebra is considered as a *contraction* of $\mathfrak{osp}\,(32|1)$, the result lacks the 55 generators of the Lorentz automorphism piece.

[28] An abelian semigroup is a set provided with a commutative product which is closed and associative. It does not need to have an identity element or inverse.

[29] Here $\mathbb{1}$ denotes the 32×32 identity matrix, whence $\mathrm{Tr}\,(\mathbb{1}) = 32$.